Tutorial 1

Week of September 10, 2018

1. Simplify the following.

   (a) **Method 1: Expanding**

   $$\frac{(3+h)^2-9}{h} = \frac{9+6h+h^2-9}{h} = \frac{h(6+h)}{h} = 6+h \quad (\text{for } h \neq 0)$$

   Method 2: Factoring as Difference of Squares

   $$\frac{(3+h)^2-9}{h} = \frac{(3+h-3)(3+h+3)}{h} = \frac{h(6+h)}{h} = 6+h \quad (\text{for } h \neq 0)$$

   (b) $$\frac{(m(x+h)+b)-(mx+b)}{h} = \frac{mx+mh+b-mx-b}{h} = \frac{mh}{h} = m \quad (\text{for } h \neq 0)$$

   (c) $$\frac{3\sqrt{2}+2\sqrt{3}}{\sqrt{12}-\sqrt{8}} = \frac{3\sqrt{2}+2\sqrt{3}}{2\sqrt{3}-2\sqrt{2}} = \left(\frac{3\sqrt{2}+2\sqrt{3}}{2\sqrt{3}-2\sqrt{2}}\right)\cdot\left(\frac{2\sqrt{3}+2\sqrt{2}}{2\sqrt{3}+2\sqrt{2}}\right) = \frac{6\sqrt{6}+12+12+4\sqrt{6}}{12-8} = \frac{10\sqrt{6}+24}{4} = \frac{5\sqrt{6}+12}{2} = \frac{5\sqrt{6}}{2}+6$$

2. Are the following always true? If not, provide a counterexample.

   (a) $\dfrac{x^2}{x+a} = \dfrac{x}{1+a}$

   This is not always true (assuming that a is a *fixed* constant). This is a common mistake people tend to make when they want to cancel the common factor of x in the numerator and denominator. If we let $x = 3$, clearly the left side does not equal the right side. The proper way of getting rid of the common factor of x is:

$$\frac{x^2}{x+a} = \frac{x}{1+\frac{a}{x}} \qquad \text{(assuming } x \neq 0\text{)}$$

   (b) $\dfrac{y}{x+y} = 1 - \dfrac{x}{x+y}$

   This is true!

$$\begin{aligned}
\frac{y}{x+y} &= \frac{y+(x-x)}{x+y} \\
&= \frac{(x+y)-x}{x+y} \\
&= \frac{x+y}{x+y} - \frac{x}{x+y} \\
&= 1 - \frac{x}{x+y} \qquad \text{(assuming } x+y \neq 0\text{)}
\end{aligned}$$

3. State the domain for each function.

   (a) $f(x) = \sqrt{x^2 - 4x}$

   The square root function has problems when the inner function $x^2 - 4x$ is less than zero. So we want the inner function to be greater or equal to zero. We set up our inequality as:

$$\begin{aligned}
x^2 - 4x &\geq 0 \\
x(x-4) &\geq 0
\end{aligned}$$

   The Inequality Method

   In order to get a product that is greater or equal to zero, we have two cases to consider:

   1. $x \geq 0$ and $x - 4 \geq 0$
   2. $x \leq 0$ and $x - 4 \leq 0$

   Case 1

   i. $x \geq 0$ (nothing to do here)

   ii. $x - 4 \geq 0 \quad \Longrightarrow \quad x \geq 4$

   The only way that x can be greater than 0 and greater than 4 at the same time is if x is greater than 4.

Case 2

i. $x \leq 0$ (nothing to do here)

ii. $x - 4 \leq 0 \quad \Longrightarrow \quad x \leq 4$

The only way that x can be less than 0 and less than 4 at the same time is if x is less than 0.

Combining the results of these two cases, we can write the domain as:

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \leq 0, x \geq 4\}$$

The Chart Method

	$\longleftarrow\!\!\!-\!\!\!-\!\!\!-\, 0$	$-\!\!\!-\!\!\!-\!\!\!-$	$4 \,-\!\!\!-\!\!\!-\!\!\!\longrightarrow$
x	$-$	$+$	$+$
$x-4$	$-$	$-$	$+$
$x(x-4)$	$+$	$-$	$+$

We notice in the middle column when $0 < x < 4$ and $0 < x - 4 < 4$, the product of the two factors is negative and we cannot take the square root of a negative quantity. Of course, when either $x = 0$ or $x = 4$, we have a product of zero which the square root function will accept. We obtain the domain as:

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \leq 0, x \geq 4\}$$

(b) $g(x) = \dfrac{x^2 + 4}{x^2 - 9}$

We run into problems when the denominator equals zero. Our inequality is:

$$x^2 - 9 = 0 \quad \Longrightarrow \quad (x-3)(x+3) = 0$$

Solving, we obtain $x = 3$ and $x = -3$. These are the points that make the denominator zero and are the points that our function should avoid. The domain is:

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \neq 3, x \neq -3\}$$

(c) $k(u) = \dfrac{u+1}{1 + \frac{1}{u+1}}$

Our function encounters problems when the denominator of $\frac{1}{u+1}$ is zero and also when $1 + \frac{1}{u+1}$ is zero.

i) $u + 1 = 0$ when $u = -1$. In order for our function to be defined, $u \neq -1$.

ii)
$$\begin{aligned} 1 + \frac{1}{u+1} &= 0 \\ (u+1) + \frac{u+1}{u+1} &= 0 \quad \text{[Multiplying throughout by } (u+1)] \\ (u+1) + 1 &= 0 \quad \text{[Assumed } (u+1) \neq 0\text{, as shown in i)]} \\ u &= -2 \end{aligned}$$

$u \neq -2$ in order for our function to be defined.

We can write our domain as:

$$\mathcal{D} = \{x \in \mathbb{R} \mid x \neq -1, x \neq -2\}$$

4. Complete the square to find the vertex. State the interval of increase and decrease.
   $y = 2x^2 + 10x + 6$

   - recall vertex form is $y = a(x-h)^2 + k$ where the vertex is at (h, k)
   - recall that $(x+c)^2 = x^2 + 2cx + c^2$ for some fixed constant c

   $$y = 2x^2 + 10x + 6 = 2(x^2 + 5x + 3)$$

   Using $x^2 + 5x + 3$, we equate the linear term's coefficient to that of the general expansion. We obtain that $5 = 2c \implies c = \frac{5}{2}$. Then $c^2 = \frac{25}{4}$.

   $$\begin{aligned} y &= 2(x^2 + 5x + 3) \\ &= 2\left(x^2 + 5x + \frac{25}{4} - \frac{25}{4} + 3\right) \\ &= 2\left(x^2 + 5x + \frac{25}{4}\right) - \frac{25}{2} + 6 \\ &= 2\left(x + \frac{5}{2}\right)^2 - \frac{13}{2} \end{aligned}$$

   Now that our equation is in vertex form, it tells us that our vertex is located at $\left(-\frac{5}{2}, -\frac{13}{2}\right)$. This parabola clearly opens upwards since the coefficient of the squared term is positive. From this, we instantly know that our function is decreasing for x values left of the vertex, and our function is increasing for x values right of the vertex.

   The interval of decrease is $\left(-\infty, -\frac{5}{2}\right]$. The interval of increase is $\left[-\frac{5}{2}, \infty\right)$.

5. Sketch the following function:

   $$y = ||x-3| - 2|$$

   Consider this function as a composition of functions where:

   - $f(x) = |x|$
   - $g(x) = x - 3$
   - $h(x) = x - 2$

   Then $y = f\left(h\left(f\left(g(x)\right)\right)\right)$.

   Go to `https://www.desmos.com/calculator/9dxwzxu0sg` to view each of these plots.

   Click the circles next to the formula to show/hide each graph. View plots in pairs to observe the effect that each function has on the previous plot!