# Tutorial 4

Week of October 1, 2018

1. Evaluate the following limits, if they exist.

(a) $\lim_{h \to 0} \frac{(-5+h)^2 - 25}{h}$

$$\frac{(-5+h)^2-25}{h} = \frac{(-5+h-5)(-5+h+5)}{h}$$
$$= \frac{(-10+h)\cdot h}{h}$$
$$= -10+h$$

$$\lim_{h\to 0}(-10+h) = -10$$

(b) $\lim_{u \to 2} \frac{\sqrt{4u+1}-3}{u-2}$

$$\frac{\sqrt{4u+1}-3}{u-2} = \frac{\sqrt{4u+1}-3}{u-2} \cdot \frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}$$
$$= \frac{4u+1-9}{(u-2)(\sqrt{4u+1}+3)} = \frac{4u-8}{(u-2)(\sqrt{4u+1}+3)} = \frac{4(u-2)}{(u-2)(\sqrt{4u+1}+3)}$$
$$= \frac{4}{\sqrt{4u+1}+3}$$

$$\lim_{u\to 2} \frac{4}{\sqrt{4u+1}+3} = \frac{4}{6} = \frac{2}{3}$$

(c) $\lim_{t \to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

$$\frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}$$
$$= \frac{1+t-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} = \frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})}$$
$$= \frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$

$$\lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}} = \frac{2}{2} = 1$$

(d) $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

$$\frac{1}{t}-\frac{1}{t^2+t}=\frac{1}{t}\cdot\frac{t+1}{t+1}-\frac{1}{t(t+1)}$$
$$=\frac{t+1-1}{t(t+1)}=\frac{t}{t(t+1)}=\frac{1}{t+1}$$

$$\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{1}=1$$

(e) $\lim_{x\to -2}\frac{2-|x|}{2+x}$

Recall that $|x|=-x$ when $x<0$, which is our case since our limit is $x\to -2$.

$$\frac{2-|x|}{2+x}=\frac{2-(-x)}{2+x}=\frac{2+x}{2+x}=1$$

$$\lim_{x\to -2}1=1$$

2. Explain why $f(x)$ is discontinuous at $x=0$. Sketch the graph of the function.

$$f(x)=\begin{cases}\cos(x) & x<0\\ 0 & x=0\\ 1-x^2 & x>0\end{cases}$$

Suppose that $f(x)$ was continuous at $x=0$. Then we require:

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}f(x)=L=f(0)$$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\cos(x)=1$$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}1-x^2=1$$

Our left limit and right limits exist and are equal to a common value of 1. However, $f(0)=0$ by the definition of this piecewise function. Since the limit as $x$ approaches 0 does not equal our function's value at $x=0$, the function is not continuous at $x=0$.

3. Find $a$ and $b$ such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x < 2 \\ ax^2 - bx + 3 & 2 \leq x < 3 \\ 2x - a + b & x \geq 3 \end{cases}$$

Our points of interest are at $x = 2$ and $x = 3$.

i.) For $f(x)$ to be continuous at $x = 2$, we require:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = L_1 = f(2)$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2^-} (x+2) = 4$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3$$

We want the right limit to equal $L_1 = f(2) = 4$.
$4a - 2b + 3 = 4$
$\implies 4a - 2b = 1$ 1

ii.) For $f(x)$ to be continuous at $x = 3$, we require:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = L_2 = f(3)$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b = 6 - a + b$$

We don't know what $L_2$ is actually equal to, but we do know that we want our left and right limits to both be equal to $L_2$. Setting left and right limits equal to one another:

$9a - 3b + 3 = 6 - a + b$

$\implies 10a - 4b = 3$ 2

Now we have a system of equations that is solvable (two equations, two unknowns). We proceed with elimination to solve for $a$ and $b$.

$$\Big(1 \times (-2)\Big) + 2$$

$$\begin{array}{ccccc} & -8a & +4b & = & -2 \\ + & 10a & -4b & = & 3 \\ \hline & 2a & & = & 1 \end{array}$$

$2a = 1 \implies a = \dfrac{1}{2}$. Plugging this back into 1 and solving for $b$, we get $b = \dfrac{1}{2}$.

Therefore our function $f(x)$ should be defined as:

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x < 2 \\ \dfrac{1}{2}x^2 - \dfrac{1}{2}x + 3 & 2 \leq x < 3 \\ 2x & x \geq 3 \end{cases}$$

For $x \geq 3$, the function is actually $2x - \frac{1}{2} + \frac{1}{2}$ but the constants cancel. We can verify that this function is continuous at $x = 2$ and $x = 3$ by recomputing our limits and function values since we now know $a$ and $b$.

4. Find the following limit or show it does not exist.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x-2}{2x+1}$

$$= \lim_{x\to\infty} \frac{3x-2}{2x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3 - 2\left(\frac{1}{x}\right)}{2 + 1\left(\frac{1}{x}\right)} = \frac{3}{2}$$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x-2}{x^2+1}$

$$= \lim_{x\to-\infty} \frac{x-2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x} - 2\left(\frac{1}{x^2}\right)}{1 + \frac{1}{x^2}} = \frac{0-0}{1+0} = 0$$

(c) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$

For $x < 0$, $x^3 = -\sqrt{x^6}$. Then $\frac{1}{x^3} = -\frac{1}{\sqrt{x^6}}$

$$= \lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to-\infty} -\frac{\frac{\sqrt{1+4x^6}}{\sqrt{x^6}}}{2\left(\frac{1}{x^3}\right) - 1} = \lim_{x\to-\infty} -\frac{\sqrt{\frac{1+4x^6}{x^6}}}{2\left(\frac{1}{x^3}\right) - 1} = \lim_{x\to-\infty} -\frac{\sqrt{\frac{1}{x^6} + 4}}{2\left(\frac{1}{x^3}\right) - 1}$$

$$= -\frac{\sqrt{0+4}}{0-1} = \frac{-2}{-1} = 2$$

(d) $\displaystyle\lim_{x\to\infty} \left(\ln(1+x^2) - \ln(1+x)\right)$

$$= \lim_{x\to\infty} \ln\left(\frac{1+x^2}{1+x}\right) = \lim_{x\to\infty} \ln\left(\frac{1+x^2}{1+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}\right) = \lim_{x\to\infty} \ln\left(\frac{\frac{1}{x}+x}{\frac{1}{x}+1}\right) = \ln\left(\lim_{x\to\infty} x\right)$$

Since the logarithm function is an increasing function, as $x \to \infty$, $\ln(x) \to \infty$. Therefore the limit does not exist.